Scattering of a Polarized Photon by a Polarized Electron

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The scattering of a polarized photon by a polarized free electron is completely calculated in covariant form using Clifford's alternating vector algebra. The result is compared with several special results of the existing literature.

Introduction

The scattering of a polarized photon by a polarized electron has completely been calculated only for the primary electron at rest in a three-dimensional vector form [1, 2]; especially the spin of the secondary electron has not been expressed in a covariant way. In consequence of that, the formulae get a compact form only by the introduction of suitably chosen mirror spins. If the result is explicitly expressed by the original spins and by Stokes parameters, the formulae become clumsy and unelegant [2]. Therefore it seems useful to recalculate the scattering in a covariant form. As will be shown, this can be done in a rather convenient way by using consistently Clifford's alternating algebra and by a suitable choice of the electromagnetic gauge.

§ 1. Scattering Formula

The essential part of the differential cross-section for Compton effect is given by

$$X_c = |\bar{u}(p_2, \sigma_2)\, F_c\, u(p_1, \sigma_1)|^2 \qquad (1.1)$$

where

$$F_c = a_2^* \frac{\mu + p_1 + k_1}{p_1 \cdot k_1} a_1 + a_1 \frac{-\mu - p_1 + k_2}{p_1 \cdot k_2} a_2^* . \qquad (1.2)$$

For reference see f. i. Akhieser-Berestetskii [3], § 25. p_1, p_2 are the four-momenta of the electrons, k_1, k_2 those of the photons; σ_1, σ_2, a_1, a_2 are the four-vectors of polarization of the four particles [4]. We take the square of a time-like vector positive, of a space-like one negative. For the propagation vectors we have

$$p_1 \cdot p_1 = p_2 \cdot p_2 = \mu^2; \quad (\mu \equiv m\, c); \quad k_1 \cdot k_1 = k_2 \cdot k_2 = 0 . \qquad (1.3)$$

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The four-vectors of polarization are orthogonal to the respective propagation vectors:

$$p_1 \cdot \sigma_1 = p_2 \cdot \sigma_2 = k_1 \cdot a_1 = k_2 \cdot a_2 = 0 . \qquad (1.4)$$

Therefore they all are space-like, and as unit vectors they have the squares

$$\sigma_1 \cdot \sigma_1 = \sigma_2 \cdot \sigma_2 = a_1^* \cdot a_1 = a_2^* \cdot a_2 = -1 . \qquad (1.5)$$

Conservation of four-momentum implies a twofold expression for the momentum transfer K from the electron- to the photon-system:

$$K = p_1 - p_2 = k_2 - k_1 . \qquad (1.6)$$

For the products of vectors we use the convention: dot for scalar product, no dot for Clifford multiplication (see § 2). — For the Dirac amplitudes we have the eigenvalue equations

$$(p - \mu)\, u(p, \sigma) = 0 ; \quad \bar{u}(p, \sigma)\,(p - \mu) = 0 ; \qquad (1.7)$$

$$(\sigma \gamma_5 - 1)\, u(p, \sigma) = 0 ; \quad \bar{u}(p, \sigma)\,(\sigma \gamma_5 - 1) = 0 . \qquad (1.8)$$

$\sigma \gamma_5$ is the pseudovector of spin, $\sigma\, p\, \gamma_5 / \mu$ the antisymmetric spin tensor. As a consequence of the eigenvalue equations and of $\bar{u}\, \gamma_0 = u^*$ the bi-spinor $\pm u\, \bar{u}$ is the projector:

$$\pm u(p, \sigma)\, \bar{u}(p, \sigma) = \frac{\mu + p}{2\, \mu} \frac{1 + \sigma \gamma_5}{2} . \qquad (1.9)$$

§ 2. Clifford Algebra

The Clifford vector algebra, of which the Dirac algebra is the four-dimensional special case, is an associative algebra of alternating products of vectors, for which we use the symbol

$$\{v_1\, v_2 \ldots v_m\} . \qquad (2.1)$$

The associative product of two alternating products is given by [5]

$$\{v_1 \ldots v_m\}\, \{w_1 \ldots w_n\} = \{v_1 \ldots v_m\, w_1 \ldots w_n\} + \text{contractions} . \qquad (2.2)$$

The contractions are meant in the sense of the ordering theorem[6] for alternating quantities, the contraction of two vectors being the scalar product:

$$\text{contr}\,\{v\,w\} = v\cdot w\,. \tag{2.3}$$

The simplest special case of Eq. (2.2) is the Clifford product of two vectors:

$$v\,w = \{v\,w\} + v\cdot w\,. \tag{2.4}$$

For the unit vectors $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ this yields the wellknown Dirac rules: $\gamma_j\,\gamma_k$ alternating for $j \neq k$; ± 1 for $j = k$ timelike resp. spacelike. – Two further special cases of Eq. (2.2) are

$$\{v\,w\}\,x = \{v\,w\,x\} + (v\,w - w\,v)\cdot x\,; \tag{2.5}$$

$$\{v\,w\}\,\{x\,y\} = \{v\,w\,x\,y\} + \{(v\,w - w\,v)\cdot(x\,y - y\,x)\} + (v\,w - w\,v)\,..\,x\,y\,. \tag{2.6}$$

In fourdimensional space we have alternating products only up to four factors. Any four-factor alternating product is proportional to the four-dimensional unit volume characterized by the quantity

$$\gamma_5 \equiv i\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3 = i\,\{\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\}\,; \quad \gamma_5\,\gamma_5 = 1\,. \tag{2.7}$$

In consequence the following expression is a pure number

$$\{v\,w\,x\,y\}\,\gamma_5 = \frac{1}{i}\,[v\,w\,x\,y]\,. \tag{2.8}$$

The last symbol means the determinant

$$[v\,w\,x\,y] = \begin{vmatrix} v_0 & v_1 & v_2 & v_3 \\ w_0 & w_1 & w_2 & w_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{vmatrix} \tag{2.9}$$

$$= v_0[\boldsymbol{w\,x\,y}] - w_0[\boldsymbol{v\,x\,y}] + x_0[\boldsymbol{v\,w\,y}] - y_0[\boldsymbol{v\,w\,x}]\,.$$

For practical calculations the following formula for n linearly dependent vectors $v_1 \dots v_n$ is important.

$$v_n\,\{v_1 \dots v_{n-1}\} = v_1\,\{v_n\,v_2 \dots v_{n-1}\} + v_2\,\{v_1\,v_n \dots v_{n-1}\} + \dots v_{n-1}\,\{v_1 \dots v_{n-2}\,v_n\}\,. \tag{2.10}$$

The validity of this formula is based on the fact that the difference of the left and right hand sides is alternating in all the n dependent vectors. The symbol $\{\,\}$ hereby can indicate any alternating expression in the $n-1$ vectors.

§ 3. Description of Propagation and Polarizations

We shall describe the dynamics of the scattering process by three vectors K, k, p; the momentum transfer K is already defined by Eq. (1.6), k and p are the internal momenta of the photon- resp. electron-systems:

$$k \equiv k_2 + k_1\,; \quad p \equiv p_1 + p_2\,. \tag{3.1}$$

The squares of these three vectors are related by

$$k^2 = -K^2 = p^2 - 4\,\mu^2\,. \tag{3.2}$$

The two internal momenta are perpendicular to K

$$k\cdot K = 0\,; \quad p\cdot K = 0\,. \tag{3.3}$$

The inversion of Eqs. (1.6) and (3.1) is

$$k_1 = \frac{k-K}{2}\,; \quad k_2 = \frac{k+K}{2}\,;$$

$$p_1 = \frac{p+K}{2}\,; \quad p_2 = \frac{p-K}{2}\,. \tag{3.4}$$

For the spin unit vectors we have the relations

$$p\cdot\sigma_1 = -K\cdot\sigma_1\,; \quad p\cdot\sigma_2 = K\cdot\sigma_2\,. \tag{3.5}$$

In order to introduce Stokes parameters to describe the polarization of a photon (see Akhieser, Berestetskii[3], § 2) we have to choose two perpendicular polarization unit vectors α, β to arrive at the following dyadic form of the light polarization density matrix:

$$-\varrho = \overline{a\,a^*} = \frac{1+\xi}{2}\,\alpha\,\alpha + \frac{1-\xi}{2}\,\beta\,\beta + \frac{\eta+i\,\zeta}{2}\,\alpha\,\beta + \frac{\eta-i\,\zeta}{2}\,\beta\,\alpha\,. \tag{3.6}$$

To make the description simple and independent of frame, there is only one choice for the unit vectors α, β: We take both of them perpendicular to both the propagation vectors k_1, k_2, and one of them – say α – in addition perpendicular to p:

$$\alpha^2 = -1\,; \quad \beta^2 = -1\,; \quad \alpha\cdot k_j = 0\,; \quad \beta\cdot k_j = 0\,; \quad \alpha\cdot p = 0\,. \tag{3.7}$$

As a consequence of momentum conservation α is also perpendicular to p_1 and p_2.

α and β are polarization vectors of both the primary and the secondary photon. They are space vectors in the "brick-wall system" of Breit; in this frame the two photons have equal frequencies and propagate into opposite directions. – By (3.7) the unit vectors α and β are determined apart from sign. We fix the signs too by demanding explicitly

$$\beta = \frac{p - k\,k\cdot p/k^2}{\sqrt{(p\cdot k)^2/k^2 - p^2}}\,; \quad [K\,k\,\alpha\,\beta] = K^2\,. \tag{3.8}$$

The two-dimensional polarization plane α, β is perpendicular to the light propagation plane K, k. The unit matrix of fourdimensional space can be expressed in dyadic form by the four perpendicular unit vectors

$$I = \frac{k\,k}{k^2} + \frac{K\,K}{K^2} - \alpha\,\alpha - \beta\,\beta\,. \qquad (3.9)$$

From (3.8) we get for an arbitrary vector v:

$$v\cdot\alpha = \frac{1}{k^2}\,[K\,k\,v\,\beta]\,. \qquad (3.10)$$

For two vectors v, w we get

$$v\,w\cdot\cdot(\alpha\beta - \beta\,\alpha) = \frac{1}{k^2}\,[K\,k\,v\,w]\,. \qquad (3.11)$$

This equation follows directly from (3.8); to get it from (3.10), one has to apply Eq. (2.10) to three dependent vectors of the polarization plane.

The inversion of Eq. (3.6) reads

$$\xi = (\alpha\,\alpha - \beta\,\beta)\cdot\cdot\overline{a\,a^*}; \quad \eta = (\alpha\beta + \beta\,\alpha)\cdot\cdot\overline{a\,a^*}; \quad \zeta = i(\alpha\beta - \beta\,\alpha)\cdot\cdot\overline{a\,a^*}\,. \qquad (3.12)$$

These expressions are independent of the gauge of a, as the principal unit vectors α, β are perpendicular to both propagation vectors k_1, k_2. By virtue of Eqs. (3.8) – (3.11) we have explicitly

$$\xi = \left(-I + \frac{k\,k - K\,K}{k^2} + 2\,\frac{\left(p - \frac{p\cdot k}{k^2}\,k\right)\left(p - \frac{p\cdot k}{k^2}\,k\right)}{p^2 - (p\cdot k)^2/k^2}\right)\cdot\cdot\overline{a\,a^*};$$

$$\eta = \frac{1}{p^2\,k^2 - (p\cdot k)^2}\left[K\,k\,p\left(p - \frac{p\cdot k}{k^2}\,k\right)\cdot\left(\overline{a\,a^*} + \overline{a^*\,a}\right)\right]; \qquad \zeta = \frac{i}{k^2}\,[K\,k\,\overline{a\,a^*}]\,. \qquad (3.13)$$

The helicity parameter ζ is for photon 1 positive for positive helicity; for photon 2 the sign is opposite.

The expression F of Eq. (1.2) is gauge-independent (it does not change if a multiple of k_j is added to a_j). This can be used to simplify the algebra by taking the polarization vectors perpendicular to one of the electron propagation vectors, e. g. p_1:

$$a_1\cdot p_1 = a_2\cdot p_1 = 0\,. \qquad (3.14)$$

This does not modify the Stokes parameters as defined by (3.12), but it alters the density matrix of each of the photon polarizations into

$$-\varrho_j = \overline{a_j\,a_j^*} = \frac{1+\xi_j}{2}\,\alpha\,\alpha + \frac{1-\xi_j}{2}\,\beta_j\,\beta_j + \frac{\eta_j + i\,\zeta_j}{2}\,\alpha\,\beta_j + \frac{\eta_j - i\,\zeta_j}{2}\,\beta_j\,\alpha\,. \qquad (3.15)$$

As α is already perpendicular to p_1, only the vector β has to be changed by a gauge transformation:

$$\beta_j = \left(I - \frac{k_j\,p_1}{k_j\cdot p_1}\right)\cdot\beta\,; \qquad (3.16)$$

§ 4. Preparing the Scattering Formula

The only easy way to evaluate the expression F of Eq. (1.2) seems following step by step the original 1938 [1] paper. Firstly we use the gauge (3.14); as this violates reciprocity, we keep in mind that the final formula has to be invariant with respect to the following reciprocity transformation

$$p \to p\,; \quad k \to k\,; \quad K \to -K\,; \quad \sigma_1 \longleftrightarrow \sigma_2\,; \quad a_1 \longleftrightarrow a_2\,. \qquad (4.1)$$

By virtue of the gauge chosen, a_j and p_1 alternate. Therefore because of the right hand factor u_1 in (1.1) the terms $\mu + p_1$ cancel out, and we are left with

$$-p_1\cdot k_1\,p_1\cdot k_2\,F = a_2{}^*\cdot a_1\,p_1\cdot(k_2\,k_1 + k_1\,k_2) + \{a_2{}^*\,a_1\}\,p_1\cdot(k_2\,k_1 - k_1\,k_2)\,. \qquad (4.2)$$

By means of Eq. (3.4) we get

$$-4\,p_1\cdot k_1\,p_1\cdot k_2\,F = a_2{}^*\,a_1(p\cdot k\,k + k^2\,K) + \{a_2{}^*\,a_1\}(k^2\,k + p\cdot k\,K)\,. \qquad (4.3)$$

We now observe that by virtue of Eq. (1.7)

$$\bar{u}_2\,K\,u_1 = \bar{u}_2(p_1 - p_2)\,u_1 = \bar{u}_2(\mu - \mu)\,u_1 = 0\,.$$

Therefore an arbitrary multiple of K can be added to (4.3), so we can substitute

$$-4\,p_1\cdot k_1\,p_1\cdot k_2\,F = \left(\frac{p\cdot k}{k^2}\,a_2{}^*\cdot a_1 + \{a_1\,a_2{}^*\}\right)Q\,; \qquad (4.4)$$

where

$$Q \equiv k^2\,k + p\cdot k\,K = 4\,p_1\cdot(k_1\,k_2 - k_2\,k_1)\,. \qquad (4.5)$$

The vector Q is perpendicular to p_1, therefore we can transform [use (1.9)]:

$$Q\,u_1\,\bar{u}_1\,Q = \frac{Q^2}{4}\left(1-\frac{p_1}{\mu}\right)(1+s_1\,\gamma_5)\,; \quad (4.6)$$

where s_1 is a "mirror-spin":

$$s_1 = \sigma_1 - 2\,\sigma_1\cdot\frac{Q\,Q}{Q^2}\,;\quad s_1\cdot s_1 = -1\,;\quad s_1\cdot p_1 = 0\,. \quad (4.7)$$

This leads to the following expression for X

$$X = \frac{4}{k^2\,[k^4-(p\cdot k)^2]}\,\bar{u}_2\left(1-\frac{p_1}{\mu}\right)G(1+s_1\,\gamma_5)\,\overline{G}\,u_2\,, \quad (4.8)$$

where

$$G \equiv p\cdot k\,a_1\cdot a_2{}^* + k^2\,\{a_1\,a_2{}^*\}\,,$$
$$\overline{G} = p\cdot k\,a_2\cdot a_1{}^* + k^2\,\{a_2\,a_1{}^*\}\,. \quad (4.9)$$

p_1 commutes with all three quantities G, $\overline{G}$, $s_1\gamma_5$, therefore in expression (4.8) the factor $1-p_1/\mu$ can be shifted freely.

§ 5. Evaluation of the Scattering Formula

By straigthforward application of Clifford multiplication (2.2) we get:

$$\begin{aligned} G(1+s_1\,\gamma_5)\overline{G} = {} & [(p\cdot k)^2\,a_1\cdot a_2{}^*\,a_2\cdot a_1{}^* + k^4(1-a_1{}^*\cdot a_2{}^*\,a_2\cdot a_1)]\,(1+s_1\,\gamma_5) \\ & + p\cdot k\,k^2\begin{pmatrix} \{(1+s_1\,\gamma_5)\,(a_2\,a_2{}^*\cdot a_1\,a_1{}^* + a_1\,a_1{}^*\cdot a_2\,a_2{}^*)\} \\ + s_1\cdot(a_2\,a_1{}^* - a_1{}^*\,a_2)\,a_1\cdot a_2{}^*\,\gamma_5 + s_1\cdot(a_2{}^*\,a_1 - a_1\,a_2{}^*)\,a_2\cdot a_1{}^*\,\gamma_5 \end{pmatrix} \\ & + k^4\,s_1\cdot(a_1{}^*\,a_2 - a_2\,a_1{}^*)\cdot(a_1\,a_2{}^* - a_2{}^*\,a_1)\gamma_5 + k^4(a_1{}^*\,a_2 - a_2\,a_1{}^*)\cdot(a_1\,a_2{}^* - a_2{}^*\,a_1)\cdot s_1\,\gamma_5 \\ & + k^4\,\{(a_1\,a_2{}^* - a_2{}^*\,a_1)\cdot(a_2\,a_1{}^* - a_1{}^*\,a_2)\,(1-s_1\,\gamma_5)\}\,. \end{aligned} \quad (5.1)$$

If we insert that into Eq. (4.8), the factor $(1-p_1/\mu)$ just enters all the alternating products of (5.1), as all vectors of Eq. (5.1) are perpendicular to p_1. So the only thing to do in order to get X is the evaluation of expressions $\bar{u}_2\{\ldots\}\,u_2$, where $\{\ldots\}$ indicates any alternating product of vectors. These expressions are traces of the product of $\{\ldots\}$ and $u_2\,\bar{u}_2$ as given by Equation (1.9). Now all these traces are evaluated easily by observing that $u_2\,\bar{u}_2$ has trace 1, and right- and left-hand eigenvalues 1 for vector p_2/μ, pseudovector $\sigma_2\,\gamma_5$, tensor $p_2\,\sigma_2\,\gamma_5/\mu$. It follows that $\bar{u}_2\{\ldots\}\,u_2$ may be $\neq 0$ only if $\{\ldots\}$ commutes with all these three quantities. This yields the following traces for scalar and pseudoscalar quantities:

$$\bar{u}_2\,u_2 = 1\,;\quad \bar{u}_2\,\gamma_5\,u_2 = 0\,. \quad (5.2)$$

For vectors we have

$$\bar{u}_2\,a\,u_2 = (p_2/\mu)\cdot a\,;\quad \bar{u}_2\,\{a\,b\,c\}\,\gamma_5\,u_2 = (1/i\,\mu)\,[p_2\,a\,b\,c]\,. \quad (5.3)$$

For pseudovectors the result is

$$\bar{u}_2\,a\,\gamma_5\,u_2 = -\sigma_2\cdot a\,;\quad \bar{u}_2\,\{a\,b\,c\}\,u_2 = (1/i)\,[a\,b\,c\,\sigma_2]\,. \quad (5.4)$$

Finally we get for tensors

$$\bar{u}_2\,\{a\,b\}\,u_2 = \frac{1}{\mu\,i}\,[a\,b\,p_2\,\sigma_2]\,;\quad \bar{u}_2\,\{a\,b\}\,\gamma_5\,u_2 = a\,b\cdot\cdot\,\frac{p_2\,\sigma_2 - \sigma_2\,p_2}{\mu}\,. \quad (5.5)$$

From Eqs. (4.8) and (5.1) we get in this way

$$\begin{aligned} \frac{\mu^2}{2}\,[(p\cdot k)^2 - k^4]X = {} & [(p\cdot k)^2\,a_1\cdot a_2{}^*\,a_2\cdot a_1{}^* + k^4(1-a_1{}^*\cdot a_2{}^*\,a_2\cdot a_1)]\,(1-s_1\cdot s_2) \\ & + p\cdot k\,k^2\,(a_2\,a_2{}^*\cdot a_1\,a_1{}^* + a_2{}^*\,a_2\cdot a_1{}^*\,a_1)\cdot\cdot\,(s_1\,s_2 - s_2\,s_1) \\ & - k^4\,(a_1\,a_1{}^* + a_2{}^*\,a_2 + a_1\,a_1{}^*\cdot a_2{}^*\,a_2 + a_2{}^*\,a_2\cdot a_1\,a_1{}^*)\cdot\cdot\,(s_1\,s_2 + s_2\,s_1) \\ & + \frac{2\,\mu}{i}\,p\cdot k[K(s_1+\sigma_2)\,(a_2\,a_2{}^*\cdot a_1\,a_1{}^* + a_1\,a_1{}^*\cdot a_2\,a_2{}^*)] \\ & + \frac{2\,\mu}{i}\,k^2[K(s_1-\sigma_2)\,(a_1\,a_1{}^* + a_2{}^*\,a_2 + a_1\,a_1{}^*\cdot a_2{}^*\,a_2 + a_2{}^*\,a_2\cdot a_1\,a_1{}^*)]\,. \end{aligned} \quad (5.6)$$

Here s_2 is another mirror-spin, defined by

$$s_2 \equiv \sigma_2 - (2/K^2) K K \cdot \sigma_2 ; \quad s_2 \cdot s_2 = -1 ; \quad \text{but } p_2 \cdot s_2 \neq 0! \tag{5.7}$$

The scattering formula (5.6) is quite compact and looks simple, but it has two disadvantages: First the appearance of the mirror-spins instead of the original ones, second the special gauge of the photon polarizations; therefore the formula is neither gauge-independent nor reciprocity symmetric. All this disadvantage disappears if we, in the next section, introduce Stokes parameters and eliminate the mirror-spins.

§ 6. Introduction of Stokes Parameters

We now adjust the scattering formula to arbitrary polarizations by introducing into Eq. (5.6) for $a_j a_j^*$ an average by means of Eq. (3.15), and for s_1, s_2 the explicit expressions (4.7), (5.7). In all these expressions we get denominators $4 p_1 \cdot k_1 = p \cdot k + k^2$ and $4 p_1 \cdot k_2 = p \cdot k - k^2$. It is convenient, to eliminate all these denominators by help of the abbreviation

$$y \equiv \frac{4 \mu^2 k^2}{(p \cdot k)^2 - k^4} = \frac{\mu^2 k_1 \cdot k_2}{2 p_1 \cdot k_1 \, p_1 \cdot k_2} = \frac{\mu^2 k_1 \cdot k_2}{2 p_2 \cdot k_1 \, p_2 \cdot k_2} . \tag{6.1}$$

In the rest frame of either electron 1 or electron 2, y is simply expressed by the scattering angle ϑ of the photon:

$$y = \frac{1 - \cos \vartheta}{2} \quad \text{in rest frame of } p_1 \text{ or } p_2 . \tag{6.2}$$

We now express by y all the scalar products appearing in the algebra. First we mark that

$$\frac{k^2 (p \cdot \beta)^2}{(p \cdot k)^2 - k^4} = 1 - y . \tag{6.3}$$

We have now

$$\beta_1 \cdot \beta_2 = 1 - 2y ; \tag{6.4}$$

$$s_1 \cdot s_2 = \sigma_1 \cdot (I + \Phi) \cdot \sigma_2 ;$$

$$\Phi \equiv \frac{y}{2 \mu^2} [p p + k k - (k p + p k) p \cdot k / k^2] ; \tag{6.5}$$

$$s_1 \cdot \beta_1 = \sigma_1 \cdot (y \beta + \varkappa) ; \quad s_2 \cdot \beta_2 = \sigma_2 \cdot (y \beta + \varkappa) ;$$

$$s_1 \cdot \beta_2 = \sigma_1 \cdot (y \beta - \varkappa) ; \quad s_2 \cdot \beta_1 = \sigma_2 \cdot (y \beta - \varkappa) . \tag{6.6}$$

where

$$\varkappa \equiv \frac{y}{4 \mu^2} \left(\frac{k \cdot p}{k^2} p - k \right) p \cdot \beta = \frac{p \cdot \beta}{k^2} k + (y - 1) \frac{p \cdot k}{k^2} \beta . \tag{6.7}$$

The scalar products (6.5) and (6.6) exhibit the reciprocity symmetry according to Eq. (4.1) which in the definitions of the vectors themselves is missing. — In treating the two alternating product terms it is sufficient to calculate the σ_2 part; the σ_1 part then follows from reciprocity symmetry according to (4.1), where instead of $a_1 \longleftrightarrow a_2$ we now have $\xi_1 \longleftrightarrow \xi_2$; $\eta_1 \longleftrightarrow -\eta_2$; $\zeta_1 \longleftrightarrow -\zeta_2$ [sign change because of Eqs. (3.8), (3.12)].

In course of the algebra α is easily eliminated by means of (3.9), (3.10), (3.11). The vectors $\beta, \varkappa$ then appear only in the form of the tensors $\beta \beta$, $\varkappa \varkappa$, $\varkappa \beta + \beta \varkappa$, explicitly given by:

$$(1 - y) \beta \beta = \frac{1}{2} \Phi + \frac{k k}{k^2} , \tag{6.8}$$

$$\varkappa \varkappa = (1 - y) \left(\frac{1}{2} \Phi + \frac{p p}{k^2} \right) , \tag{6.9}$$

$$p \cdot k \frac{y}{2 \mu^2} (\varkappa \beta + \beta \varkappa) = - \left(1 + \frac{k^2}{2 \mu^2} y \right) \Phi - \frac{y}{2 \mu^2} (k k + p p) . \tag{6.10}$$

To simplify the terms containing determinants some use of Eq. (2.10) has to be made. — The final result in the form we consider most simple reads:

$$\begin{aligned} 2 \mu^2 X = {} & \left(2 + \frac{k^2}{\mu^2} y \right) [1 + (1 - 2y) (\eta_1 \eta_2 \sigma_1 \cdot \sigma_2 - \zeta_1 \zeta_2) + (\eta_1 \eta_2 - \xi_1 \xi_2) \sigma_1 \cdot \Phi \cdot \sigma_2 - \xi_1 \xi_2 \sigma_1 \cdot \sigma_2] \\ & - 4 y (1 - y) (1 - \xi_1) (1 - \xi_2) (1 - \sigma_1 \cdot \sigma_2) + 2 (1 - 2y) (\zeta_1 \zeta_2 \sigma_1 \cdot \sigma_2 - \eta_1 \eta_2) - 2 \sigma_1 \cdot \Phi \cdot \sigma_2 (1 - \zeta_1 \zeta_2) \\ & + 2 \xi_1 \xi_2 - 2 \sigma_1 \cdot \sigma_2 + 2 y \left((1 - \xi_1) (1 - \xi_2) - \eta_1 \eta_2 - \zeta_1 \zeta_2 \right) \sigma_1 \cdot \Phi \cdot \sigma_2 + \frac{y^2}{\mu^2} \xi_1 \sigma_1 \cdot (k + p) (k - p) \cdot \sigma_2 \\ & + \frac{y^2}{\mu^2} \xi_2 \sigma_1 \cdot (k - p) (k + p) \cdot \sigma_2 + \frac{2 y^2}{\mu^2} \xi_1 \xi_2 \sigma_1 \cdot (p p - k k) \cdot \sigma_2 + \frac{2 y^2}{\mu^2} \eta_1 \eta_2 \sigma_1 \cdot (p p + k k) \cdot \sigma_2 \\ & + \frac{y^2}{\mu^2} \det (K p \sigma_2 \sigma_1) (\eta_1 \xi_2 - \xi_1 \eta_2) - \frac{y^2}{\mu^2 k^2} \det \left(K k p k \cdot (\sigma_1 \sigma_2 - \sigma_2 \sigma_1) \right) (\eta_1 - \eta_2) \end{aligned}$$

$$
\begin{aligned}
& -\frac{y^2}{\mu^2 k^2} \det\left(K\, k\, p\, p\cdot(\sigma_1 \sigma_2 + \sigma_2 \sigma_1)\right)(\eta_1 + \eta_2) \qquad (6.11)\\
& +\frac{y^2}{\mu^2 k^2} \det\left[K\, k\, p\left(\left(1-\frac{(p\cdot k)^2}{4\,\mu^2 k^2}\right)p + \frac{p\cdot k}{4\,\mu^2}\,k\right)\cdot(\sigma_1 \sigma_2 + \sigma_2 \sigma_1)\right](\eta_1 \xi_2 + \xi_1 \eta_2)\\
& +\frac{2\,y^2}{\mu}\, p\cdot(\sigma_2 - \sigma_1)(\zeta_1 + \zeta_2) + \frac{2\,y(1-y)}{\mu}\, k\cdot(\sigma_2 + \sigma_1)(\zeta_1 - \zeta_2) + \frac{y^2}{2\,\mu^3}\,(k\cdot p\, p - p^2 k)\\
& \cdot(\sigma_2 + \sigma_1)(\xi_1 \zeta_2 - \xi_2 \zeta_1) - \frac{y^2}{2\,\mu^3}\left(p\cdot k\, k + 4\,\mu^2 p - \frac{(p\cdot k)^2}{k^2}\,p\right)\cdot(\sigma_2 - \sigma_1)(\xi_1 \zeta_2 + \zeta_1 \xi_2)\\
& +\frac{p\cdot k}{2\,\mu^3 k^2}\, y^2 \det\left(K\, k\, p(\sigma_2 + \sigma_1)\right)(\eta_1 \zeta_2 - \zeta_1 \eta_2) + \frac{y^2}{2\,\mu^3}\det\left(K\, k\, p(\sigma_2 - \sigma_1)\right)(\eta_1 \zeta_2 + \zeta_1 \eta_2)\,.
\end{aligned}
$$

What concerns the physical meaning of the Stokes parameters herein, ξ_1, η_1, ζ_1 clearly describe the polarization state of the primary light. ξ_2, η_2, ζ_2 describe the polarization component, for which we ask; this is either a pure (linear, circular, elliptic) polarization, so $\xi_2{}^2 + \eta_2{}^2 + \zeta_2{}^2 = 1$; or no specification of polarization, that is: polarization average, $\xi_2 = \eta_2 = \zeta_2 = 0$ – in this case the majority of the terms vanishes. – Similarily we specify the spins σ_1, σ_2. Clearly σ_2 is the secondary spin direction, for which we ask, a space-like unit vector; if we are not interested in σ_2, then we average $\overline{\sigma_2} = 0$, which again makes most of the terms vanish. On the other hand, σ_1 indicates the spin (polarization) of the primary electrons; if they were fully polarized, then $\sigma_1 \cdot \sigma_1 = -1$; in reality they are at most partly polarized (ferromagnets), so we have to understand by σ_1 the average spin $(0 \leqq -\sigma_1 \cdot \sigma_1 \leqq 1)$; in case of unpolarized primary electrons $\sigma_1 = 0$.

This now is the formula for Compton-scattering. The annihilation formula only differs by a sign change of ζ_1 and by the definitions of p, k, K according to Eqs. (1.6), (3.1) and footnote [4].

The first two terms of Eq. (6.11) contain the Klein-Nishina formula. To compare with the usual form, mark that

$$2 + \frac{k^2}{\mu^2}\, y = \frac{k_1 \cdot p_1}{k_2 \cdot p_1} + \frac{k_2 \cdot p_1}{k_1 \cdot p_1}\,. \qquad (6.12)$$

If p_1 (or p_2) is at rest, this is $\omega_1/\omega_2 + \omega_2/\omega_1$. Furthermore because of Eq. (6.2) in either of the rest frames $4\,y(1-y) = \sin^2 \vartheta$.

§ 7. Check with Previous Formulas

To compare our formula with Fr [1] and LT [2] we have to introduce instead of the covariant spin-four-vectors σ space vectors, called $\boldsymbol{\zeta}$ in LT and $\boldsymbol{\sigma}$ in Fr, indicating the average spin vector in the rest system of the resp. electron. σ is expressed by

$$\sigma = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{\mu}\,,\ \boldsymbol{\zeta} + \frac{\boldsymbol{p}\,\boldsymbol{p}\cdot\boldsymbol{\zeta}}{\mu(p_0 + \mu)}\right), \qquad (6.13)$$

$(p_0, \boldsymbol{p}) = p$ being the momentum vector of the electron. Specified to primary electron at rest (6.11) now checks completely with Fr [1], and with LT, except a missing subscript 0 in LT (2.7) "$\boldsymbol{k}_0 \cos\vartheta + \boldsymbol{k}$", and a missing factor 2 of the last line of LT (2.16). Some of the LT formulas are considerably simplified when products of pseudoscalars are expressed by scalar products (Gram's determinant):

$$
\begin{aligned}
8\,\Phi_2(\boldsymbol{\zeta}^0, \boldsymbol{\zeta}) = {} & (1+\cos^2\vartheta)\,\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}^0 + (1+\cos\vartheta)\,(\boldsymbol{k}\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0 - \boldsymbol{k}^0\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}) \qquad \mathrm{LT}(2.11)\\
& + \frac{1+\cos^2\vartheta}{k^0 - k + 2}\,(\boldsymbol{k}^0 - \boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot(\boldsymbol{k}^0 - \boldsymbol{k}) - (1-\cos\vartheta)\,\frac{k^0 + k}{k^0 - k + 2}\,(\boldsymbol{k}^0 - \boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot(\boldsymbol{n}^0 - \boldsymbol{n})
\end{aligned}
$$

$$8\,\Phi_3(\xi^0,\zeta^0,\zeta) = (1-\cos\vartheta)\left[\begin{array}{l} \xi_1{}^0\left(\begin{array}{l}(1+\cos\vartheta)\,\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}^0 + \boldsymbol{n}\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{k}^0 - \boldsymbol{k}^0\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n} \\ + \dfrac{(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}}{k^0-k+2}\left((k^0-k\cos\vartheta)\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n} - (k^0\cos\vartheta-k)\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0\right)\end{array}\right) \\ + \xi_2{}^0\left(k^0(\boldsymbol{n}-\boldsymbol{n}^0\cos\vartheta)\cdot\boldsymbol{\zeta}\times\boldsymbol{\zeta}^0 + \dfrac{k^0+k}{k^0-k+2}\,(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0\times\boldsymbol{n}\right)\end{array}\right] \qquad \text{LT}(2.14)$$

$$8\,\Phi_3(\xi,\zeta^0,\zeta) = (1-\cos\vartheta)\left[\begin{array}{l} \xi_1\left(\begin{array}{l}(1+\cos\vartheta)\cdot\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}^0 + \boldsymbol{k}\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0 - \boldsymbol{n}^0\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{k} \\ + \dfrac{(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}}{k^0-k+2}\left((k^0\cos\vartheta-k)\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0 - (k^0-k\cos\vartheta)\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}\right)\end{array}\right) \\ + \xi_2\left(k(\boldsymbol{n}\cos\vartheta-\boldsymbol{n}^0)\cdot\boldsymbol{\zeta}\times\boldsymbol{\zeta}^0 + \dfrac{k^0+k}{k^0-k+2}\,(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0\times\boldsymbol{n}\right)\end{array}\right] \qquad \text{LT}(2.15)$$

$$\begin{aligned} 8\,\Phi_4(\xi^0,\xi,\zeta^0,\zeta) = {} & \boldsymbol{\zeta}\cdot\boldsymbol{\zeta}^0\left(\begin{array}{l}\xi_1{}^0\xi_1(1+\cos\vartheta)^2 - \xi_2{}^0\xi_2(k^0-k)\,(1-\cos\vartheta)^2 + (\xi_1{}^0\xi_1+\xi_2{}^0\xi_2)\,(k^0-k)(1-\cos\vartheta) \\ +2\cos\vartheta\,(-\xi_1{}^0\xi_1+\xi_2{}^0\xi_2+\xi_3{}^0\xi_3)\end{array}\right) \\ & + (\xi_1{}^0\xi_2-\xi_2{}^0\xi_1)\tfrac{1}{2}(1-\cos\vartheta)\left(\begin{array}{l}(\boldsymbol{k}^0+\boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0\times\boldsymbol{n}\,(k^0+k) - \dfrac{(k^0+k)^2}{k^0-k+2}\,(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot(\boldsymbol{n}^0\times\boldsymbol{n}) \\ -(1+\cos\vartheta)\,(\boldsymbol{k}^0+\boldsymbol{k})\cdot\boldsymbol{\zeta}\times\boldsymbol{\zeta}^0\end{array}\right) \\ & - (\xi_1{}^0\xi_2+\xi_2{}^0\xi_1)\tfrac{1}{2}(1-\cos\vartheta)^2\,(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}\times\boldsymbol{\zeta}^0 \\ & + (\xi_1{}^0\xi_1+\xi_2{}^0\xi_2)\left(\begin{array}{l}(1-\cos\vartheta)\,(\boldsymbol{n}^0\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{k}-\boldsymbol{n}\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{k}^0) + 2(\boldsymbol{k}\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0-\boldsymbol{k}^0\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}) \\ -(k^0-k)\,(\boldsymbol{n}^0-\boldsymbol{n})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot(\boldsymbol{n}^0-\boldsymbol{n})\end{array}\right) \\ & + \xi_3{}^0\xi_3(1+\cos\vartheta)\,(\boldsymbol{k}\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}^0-\boldsymbol{k}^0\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot\boldsymbol{n}) \qquad \text{LT}(2.16) \\ & + \frac{k^0-k}{k^0-k+2}\,(1+\cos\vartheta)\,(\xi_1{}^0\xi_1+\xi_2{}^0\xi_2+\xi_3{}^0\xi_3)\,(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot(\boldsymbol{n}^0+\boldsymbol{n}) \\ & - \frac{(\boldsymbol{k}^0-\boldsymbol{k})\cdot\boldsymbol{\zeta}\,\boldsymbol{\zeta}^0\cdot(\boldsymbol{k}^0-\boldsymbol{k})}{k^0-k+2}\left((1-\cos\vartheta)^2\xi_1{}^0\xi_1+2\cos\vartheta(\xi_1{}^0\xi_1+\xi_2{}^0\xi_2)+2\,\xi_3{}^0\xi_3\right). \end{aligned}$$

Landau and Lifschitz started calculating F_c [Eq. (1.2)!] up to a certain extent[7]; we carried through the completion of their idea, again achieving result (6.11).

Low[8] made the computer calculate a part of the formula for Compton scattering, omitting all terms containing determinants; his result turns out to agree with ours.

[1] W. Franz, Ann. Physik Leipzig **33**, 689 [1938]; quoted as Fr.

[2] F. W. Lipps and H. A. Tolhoek, Physica **20**, 85, 395 [1954]; quoted as LT.

[3] A. J. Akhieser, V. B. Berestetskii, Quantum Electrodynamics, Wiley & Sons, New York 1965.

[4] Pair annihilation differs from Compton scattering just by the substitution $p_2 \to -p_2$, $\sigma_2 \to -\sigma_2$, $k_1 \to -k_1$, $a_1 \to a_1^*$.

[5] Lagally-Franz, Vektorrechnung, 7. Aufl., Akadem. Verlagsgesellsch., Leipzig 1964, p. 356 ff.

[6] Jauch-Rohrlich, Theory of Photons and Electrons, Addison-Wesley, Cambridge Mass. 1955, p. 449 ff.

[7] Landau-Lifschitz: Lehrbuch der theor. Physik, IV a, 2. Auflage, Akademie-Verlag, Berlin 1971.

[8] J. C. Low, Diplomarbeit Univ. München 1970: „Polarisationsexperimente zur Prüfung des Elektron-Propagators".